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Dynamic Formulation of Coefficient of Restitution

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ABSTRACT

This paper is concerned with the preparation of a "mathematical model" involving collisions between two bodies and the subsequent analog computer program. It considers a fictitious damped spring as a coefficient of restitution "generator." Formulae are derived for spring stiffness and the viscous damping in both one and two degrees of freedom.

CONTENTS

SYMBOLISM

x_i	displacement of i th mass
$x = x_1 - x_2$	relative displacement of the masses
X	computer voltage analogous to x
V_i	velocity of i th mass just before contact
V_i'	velocity of i th mass just after separation
$\dot{x}_i$	velocity of i th mass as a function of time
$\dot{x} = (\dot{x}_1 - \dot{x}_2)$	relative velocity of the masses
$\ddot{x}_i$	acceleration of i th mass
t	time
t_1	time when $(x_1 - x_2)$ is at a maximum
t_2	time when $(x_1 - x_2) = 0$ the second time
$\mathcal{T}$	machine time
M	mass
M_1	mass of first body
M_2	mass of second body
R	damping factor (#sec/in)
K	spring stiffness (#/in)
a	coefficient of restitution
q	mass ratio

Primes denote velocity and displacement immediately after separation, i.e., relative displacement is zero.

SUBJECT

Representation of the Coefficient of Restitution in a Dynamic System with Analog Computer Applications.

OBJECTIVES

1. To determine a system of differential equations whereby the energy losses due to the coefficient of restitution may be represented as a continuous function of time.

2. To formulate an Analog Computer program which simulates the coefficient of restitution.

3. To evaluate the parameters representing the collisions between barrel extension and buffer, and barrel extension and receiver in the M-73 machine gun.

CONCLUSION

It has been analytically determined that a fictitious damped spring produces energy losses analogous to those produced by the coefficient of restitution.

Formulae have been developed to permit the evaluation of the spring stiffness and viscous damping factors as functions of the coefficient of restitution and the impact time. These formulae have been applied to the two major collisions which occur in the M-73 automatic weapon.

The Analog Computer was then used to simulate these collisions and the resulting solution was compared with test data. In comparing the results, it was found that the differences were those due to human error in interpretation of the respective recordings.

1. INTRODUCTION

a. When two bodies collide, a transfer of energy takes place through the active and reactive forces of impact. These forces are dependent on the shape and elastic properties of the bodies. As a rule, the energy before impact is greater than the energy after impact. The loss of energy in the form of heat and deformation may by expressed by the coefficient of restitution.

b. If the time of impact and the elastic deformation are not considered, the relationships[1] between the initial velocities (immediately before contact) and the final velocities (immediately after separation) are defined as follows:

$$V_1' = \frac{M_2 V_2 (1+a) + (M_1 - aM_2) V_1}{M_1 + M_2}$$

$$V_2' = \frac{M_1 V_1 (1+a) - (aM_1 - M_2) V_2}{M_1 + M_2} \qquad \text{A}$$

$$a = -\frac{(V_1' - V_2')}{(V_1 - V_2)}$$

where "a" is defined as the coefficient of restitution.

c. When solid stops or buffers must be represented by means of dynamic equations, serious errors may be introduced into the equations of motion if the collisions are assumed to be completely plastic or elastic. Therefore, it is necessary to introduce realistic energy losses.

d. Experimental results show that when two bodies collide, a deformation of the bodies occurs during contact time. The bodies may resume their original shape after contact. Consequently, the equations of motion should be in such a form as to permit slight displacements. This is easily accomplished by introducing a fictitious spring, having a high spring rate, into the system. Since it is necessary for the spring to return less energy into the system than it absorbs -- viscous damping is employed to produce the required energy losses.

[1] Superscripts denote reference number.

e. Paragraph (d) describes the applied logic used to develop the dynamic equations of motion which consider the coefficient of restitution. It is now possible to construct an analog computer patch diagram through which the system may be evaluated. Subsequent sections of this report will:

(1) Prove the validity of the applied logic;

(2) Show consistency between dynamic and static equations;

(3) Demonstrate the application of the dynamic equations of motion to analogous systems of the M-73 machine gun.

f. In the course of this analysis it is assumed that the coefficient of restitution remains constant over the entire range of velocities. This assumption is not entirely true[2], but when automatic weapons are under consideration the range of impact velocities is narrow and the change in the coefficient of restitution is trivial.

2. PROCEDURE

a. Consider two bodies moving along the same path in space at constant velocities V_1 and V_2. If the velocities are such that the leading body will not outrun the other, they will eventually collide with a resulting loss of energy. It is assumed that upon colliding, a force is exerted on the bodies which is analogous to the force developed by a spring with viscous damping.

b. Figure 1 depicts a typical two body system at the time of initial contact. Positions relative to time are given in rectangular coordinates at $t = 0$, by $x_1 = x_2 = 0$, $\dot{x}_1 = V_1$ and $\dot{x}_2 = \pm V_2$.

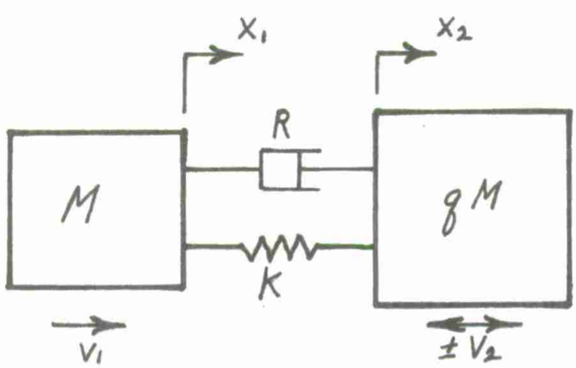

Figure 1

The damped spring will produce a force affecting the motion of the bodies only when the relative displacement (x_1-x_2) is positive. The equations of motion are as follows:

$$M\ddot{x}_1 + R(\dot{x}_1-\dot{x}_2) + k(x_1-x_2) = 0$$
$$qM\ddot{x}_2 + R(\dot{x}_2-\dot{x}_1) + k(x_2-x_1) = 0 \qquad (1)$$

Initial conditions: $t=0,\ x_1 = x_2 = 0,\ \dot{x}_1 = V_1,\ \dot{x}_2 = V_2$

The general solution of this equation is dependent on the roots of the auxiliary equation:

$$p^2 + \frac{R(1+q)}{qM}p + \frac{K}{qM}(1+q) = 0$$

This equation is similar to that obtained for one degree of freedom where the viscous damping may be such that spring may be

a. under damped $4KMq > R^2(q+1)$

b. critically damped $4KMq = R^2(q+1)$

c. over damped $4KMq < R^2(q+1)$

The over damped and critically damped cases are of minor importance in this study as they produce a coefficient of restitution of zero. They are considered as special cases in Section 2 e (2),(3).

The under damped case, which produces an oscillatory motion, is of interest because of its potential to define a coefficient of restitution in the range

$$0 < a \leq 1 .$$

Then, if, $4KMq > R^2(q+1)$, the general solution of equation (1) is:

$$X_1 = A_1 + B_1 t + e^{-\rho t}[D_1 \sin \omega t + E_1 \cos \omega t]$$
$$X_2 = A_2 + B_2 t + e^{-\rho t}[D_2 \sin \omega t + E_2 \cos \omega t] \qquad (2)$$

where $\rho = \frac{(q+1)R}{2qM}$ (2a)

and $\omega = \frac{(q+1)^{1/2}}{2qM}\sqrt{4KMq - (1+q)R^2}$. (2b)

Applying the initial conditions yields

$$X_1 = V_2 t + \frac{(V_1 - V_2)q}{(q+1)\omega}\left[\frac{\omega}{q}t + e^{-\rho t}\sin\omega t\right] \qquad (3)$$

$$X_2 = V_2 t + \frac{V_1 - V_2}{\omega(q+1)}\left[\omega t - e^{-\rho t}\sin\omega t\right]$$

To obtain the separation velocities it is necessary to solve for the time when the relative displacement becomes zero. Hence, subtracting equations (3) and equating to zero, yields

$$X = X_1 - X_2 = \frac{(V_1 - V_2)\, e^{-\rho t}}{\omega} \sin\omega t = 0 \tag{4}$$

Let t_2 be the time when $x = 0$, then

$$\sin \omega t_2 = 0 \tag{5a}$$

and

$$t_2 = \frac{\pi}{\omega}. \tag{5b}$$

Differentiating equations (3) and substituting equations (5) yield the velocities immediately after separation,

$$\dot{X}_1' = V_2 + \frac{(V_1 - V_2)q}{q+1}\left[\frac{1}{q} - e^{-\frac{\rho\pi}{\omega}}\right] \tag{6}$$

$$\dot{X}_2' = V_2 + \frac{(V_1 - V_2)}{q+1}\left[1 + e^{-\frac{\rho\pi}{\omega}}\right]$$

Subtracting to obtain the final relative velocity

$$\dot{X}' = \dot{X}_1' - \dot{X}_2' = (V_1 - V_2)\, e^{-\frac{\rho\pi}{\omega}}. \tag{7}$$

As the initial relative velocity is

$$\dot{X} = \dot{X}_1 - \dot{X}_2 \;, \tag{8}$$

then by definition, the coefficient of restitution may be obtained from (7) and (8)

$$a = \frac{-\dot{X}'}{\dot{X}} = -\frac{V_1' - V_2'}{V_1 - V_2} = e^{-\frac{\rho\pi}{\omega}} \;. \tag{9}$$

Equation 9 defines the coefficient of restitution as a function of ρ and ω which are functions of M, K, and R. It is important to note that the coefficient of restitution is completely independent of the magnitude of the velocities.

Now letting $a = \frac{1}{b}$;

then from equations (9), (2) and (2b)

$$\ln b = \rho \frac{\pi}{\omega} = \frac{\pi R \sqrt{q+1}}{\sqrt{4KMq-(1+q)R^2}} \tag{10}$$

Therefore

$$R = \ln b \sqrt{\frac{4KM}{\frac{q+1}{q}\left[\pi^2+(\ln b)^2\right]}} \tag{11}$$

and

$$\omega = \pi \sqrt{\frac{(1+q)K}{qM}\left[\frac{1}{\pi^2+(\ln b)^2}\right]} \tag{12}$$

Equations (11) and (12) define the damping factor R and the frequency of oscillation ω as a function of K, M, q and a.

c. Maximum Displacement. When two bodies collide some indentation will exist, or our fictitious spring will be compressed. It is highly desirable to have a spring rate which is realistic; therefore an idea as to the possible compression of the spring is required.

From equation (4) we have

$$x = \frac{V_1 - V_2}{\omega} e^{-\rho t} \sin \omega t \tag{4}$$

Differentiating,

$$\dot{x} = \frac{V_1 - V_2}{\omega} e^{-\rho t} \left[\omega \cos \omega t - \rho \sin \omega t\right] \tag{13}$$

If the maximum displacement occurs at $t = t_1$, then

$$\tan \omega t_1 = \frac{\omega}{\rho} = \frac{\pi}{\ln b} \frac{q}{q+1} \tag{14}$$

or

$$t_1 = \frac{\tan^{-1} \frac{q\pi}{(q+1)\ln b}}{\omega} . \tag{15}$$

Substituting $t=t_1$ into equation (4) yields:

$$X_{max} = \frac{V_1 - V_2}{\omega} e^{-\rho t} \sin\left(\tan^{-1} \frac{q\pi}{(q+1)\ln b}\right) \tag{16}$$

d. Consistency. The derived equation 1 thru 16 must be completely consistent with equation A.

If

$$a = e^{-\frac{\rho\pi}{\omega}} \tag{9}$$

is substituted into equation (6), the equations may be written as follows:

$$\dot{X}_1' = V_1' = V_2 + \frac{V_1 - V_2}{(q+1)} - \frac{a(V_1 - V_2)}{(q+1)} q \tag{6a}$$

$$\dot{X}_2' = V_2' = V_2 + \frac{V_1 - V_2}{(q+1)} + \frac{a(V_1 - V_2)}{(q+1)} . \tag{6b}$$

Substituting

$$M = M_1 = \frac{M_2}{q}$$

into equation A yields,

$$V_1' = \frac{q(1+a)V_2 + V_1(1-aq)}{q+1} \tag{17}$$

$$V_2' = \frac{V_1(1+a) - (a-q)V_2}{q+1} \tag{18}$$

When the terms on the right side of equations (6a) and (6b) are placed over a common denominator and the terms collected, the results are identical to those of equation A. It is evident that the derivations are mathematically consistent.

e. Special Cases

(1) One Degree of Freedom. The equations 1 through 16 may be readily utilized to represent a system in one degree of freedom. If the mass of the second body is assumed to be infinite and motionless, q will equal ∞ and V_2 will equal 0 ;

therefore

$$\frac{q+1}{q} \quad \text{and} \quad \frac{(q+1)^2}{q^2} = 1$$

(2) <u>Critical Damping</u>. A special case of critical damping occurs when

$$4\ K\ Mq = R^2\ (q+1)$$

The general solution of equation (1) is then

$$X_1 = A_1 + B_1 t + D_1\ e^{-\rho t} + E_1 t e^{-\rho t} \quad (19)$$

$$X_2 = A_2 + B_2 t + D_2\ e^{-\rho t} + E_2 t e^{-\rho t} \quad (19a)$$

applying initial conditions

$$x_1 = \frac{(q\ V_2 + V_1)t + (V_1 - V_2)qte^{-\rho t}}{q+1} \quad (20)$$

$$x_2 = \frac{(q\ V_2 + V_1)t - (V_1 - V_2)t\ e^{-\rho t}}{q+1} \quad (20a)$$

the relative displacement and velocity are then

$$x = x_1 - x_2 = te^{-\rho t}\left[V_1 - V_2\right] \quad (21)$$

$$\dot{x} = \dot{x}_1 - \dot{x}_2 = (V_1 - V_2)(1-\rho t)e^{-\rho t} \quad (21a)$$

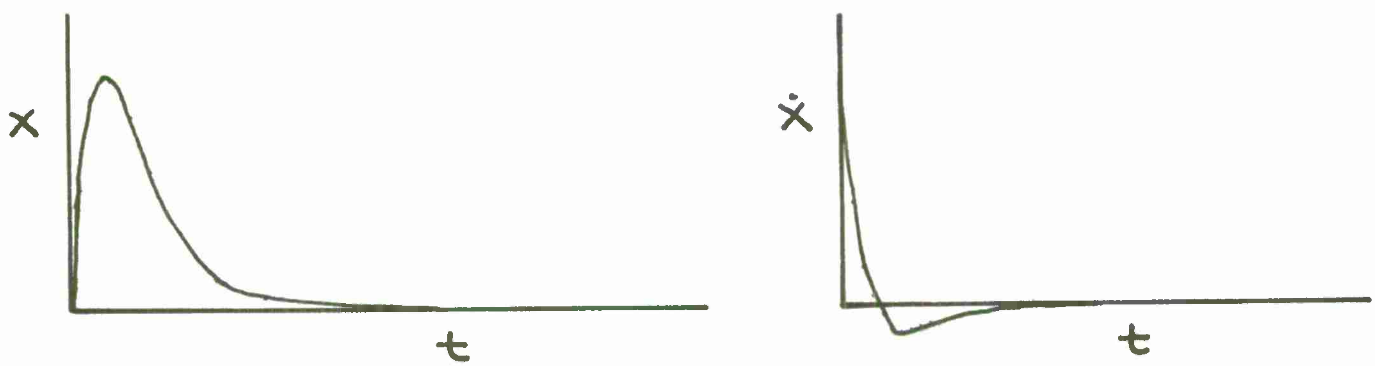

Figure 2

The relative displacement and velocity as a functions of time are shown in Figure 2. Since this is a case of critical damping, the relative displacements and velocities will be zero when $t = \infty$. It can be seen that the sign of the displacement is never changed; consequently the coefficient of restitution is zero.

(3) Over Damping. An over damped condition occurs when

$$R^2 (q+1) > 4KMq$$

The general solution of equation (1) is then

$$X_1 = A_1 + B_1 t + D_1 e^{-P_1 t} + E_1 e^{-P_2 t}$$

$$X_2 = A_2 + B_2 t + D_2 e^{-P_1 t} + E_2 e^{-P_2 t}$$

where $-P_1$ and $-P_2$ are the negative real roots of

$$P^2 + \frac{R}{qM}(1+q)P + \frac{K}{qM}(1+q) = 0$$

The over damped case, like the critically damped case, produces a coefficient of restitution equal to zero. The difference in the two cases is that the velocity of the overdamped system becomes zero at some finite time and the displacement does not approach zero.

It may readily be seen that as the relative displacement does not change sign, the two masses are lumped into one and the original masses are permanently deformed.

3. DISCUSSION

a. The derived equations may now be applied to a dynamic system to evaluate the rate and damping of a fictitious spring and the effects of the coefficient of restitution.

These equations indicate that some displacement must be tolerated. In the case of solid collisions, a very small displacement is permissible. In the case of buffers, the displacement obtained mathematically will be lower than that observed because the derived equations do not take into account the rate of the buffer spring. In either case, the small displacements during collision are not significant if the velocities and time of contact are correct.

b. M-73 Machine Gun Analysis. In the M-73 machine gun two major collision systems exist, in which the coefficient of restitution is important. The characteristic type of each is described as follows:

(1) Solid Stop. A collision between barrel extension and receiver.

(2) Buffer. A collision between barrel extension and buffer spring.

c. Time displacement curves were plotted for solid stop and buffer collisions in the M-73 machine gun on a rigid mount. These curves reflect the following conditions:

Solid Stop Collision	Buffer Collision
$q = \infty$	$q = \infty$
$M = 9.5/386 \cdot \frac{\#sec^2}{in}$	$M = 9.5/386 \quad \frac{\#sec^2}{in}$
$V = 60$ in/sec	$V = 55$ in/sec
$V' = -5$ in/sec	$V' = -30$ in/sec
$t_2 = 0.002$ sec	$t_2 = 0.00254$ sec

with $a = \frac{1}{b}$

Applying equation (10)

$b = 12$	$b = \frac{11}{6}$
$\ln b = 2.48491$	$\ln b = 0.60014$

From equations (5) and (12)

$$t_2 = \frac{\pi}{\omega} = \left(\pi^2 + (\ln b)^2\right)^{1/2} \sqrt{\frac{M}{K}}$$

$K = 98,616.25$	$K = 39,011.35$

Substituting into equation (12)

$$\omega = \pi \sqrt{\frac{K}{M[(\pi^2 + \ln b^2)^2]}}$$

$= 1.570.795$ $\qquad$ $= 1,236.846$.

Then from equation (11)

$$R = \ln b \sqrt{\frac{4\,KM}{\pi^2 + \ln b^2}} \; .$$

Solving for x_{max} from equation (16)

$$X_{max} = \frac{V}{\omega} e^{-\rho t_1} \sin\left(\tan^{-1} \frac{\pi}{\ln b}\right)$$

$t_1 = 1.20426$ $\qquad$ $t_1 = 5.18294$.

Substituting into equation (15)

$$\omega t_1 = \tan^{-1} \frac{\pi}{\ln b}$$

$t_1 = 0.57393$ m sec. $\qquad$ $t_1 = 1.11589$ m sec.

and

$\sin \omega t_1 = 0.78431$ $\qquad$ $\sin \omega t_1 = 0.98189$.

Then from equation (10)

$$\ln b = \rho \frac{\pi}{\omega}$$

$\rho = 1.24245 \times 10^3$ $\qquad$ $\rho = 0.23864 \times 10^3$

$\rho t_1 = 0.71308$ $\qquad$ $\rho t_1 = 0.26630$

$e^{-\rho t_1} = 0.49013$ $\qquad$ $e^{-\rho t_1} = 0.76621$.

Substituting back into equation (16)

$$X_{max} = \frac{V e^{-\rho t_1}}{\omega} \sin\left(\tan^{-1} \frac{\pi}{\ln b}\right)$$

$x\ max = 14.7 \times 10^{-3}$ in $\qquad$ $x\ max = 33.45 \times 10^{-3}$ in .

d. Briefly stated, the summarized results of the preceding calculations are as follows:

(1) Solid Stop Collision. With K = 98,600 #/in and R=61.1, a coefficient of restitution of 0.08333 will be maintained if a maximum displacement of 14.7×10^{-3} inches is acceptable.

(2) Buffer Collision. With K=39,000 and R=11.73 a coefficient of restitution of 0.54545 will be maintained if a maximum displacement of 33.45×10^{-3} inches is acceptable.

e. Analog Computer Application.[3,4] The analog computer program for the M-73 machine gun is such that actual time is slowed down by a factor of 200. Hence, the forces simulating losses due to the coefficient of restitution will be of very short duration.

To check the utility of the derived relationships, consider a condition during which the barrel extension, moving with free velocity, strikes the receiver and bounces back. This condition may be satisfied by the following differential equation:

$$M\ddot{x} + F_s = 0$$

with initial conditions

$$t=0 \qquad x=0.600 \qquad \dot{x} = -60 \text{ in/sec}$$

where $F_s=0$ when $x \geq 0$

and $F_s = 98{,}600x + 61.1\,\dot{x}$ when $x<0$

Substituting the values of mass and dividing

$$\ddot{x} + 2{,}485.162\dot{x} + 4{,}010{,}425.26x = 0$$

(1) Computer Equations.

if $\mathcal{T} = 200t$

$$x = 2X$$

$$\dot{x} = 400\dot{X}$$

$$\ddot{x} = 80{,}000\ddot{X}$$

then $\ddot{X} + 12.42581\dot{X} + 100.26X = 0$

when $\mathcal{T} = 0$; $X = 30.00$ volts, $\dot{X} = 15.00$ volts

f. Patch Diagram. The patch diagram illustrated in Figure 3 is used to program the analog computer for evaluation of the collision systems in the M-73 machine gun.

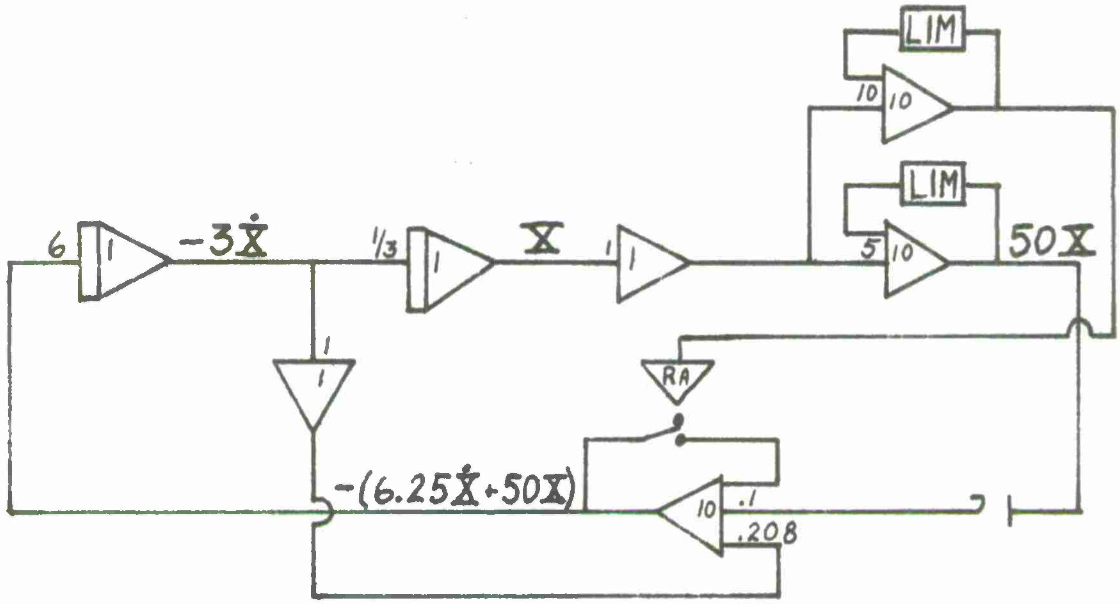

Figure 3

This computer program has been tested repeatedly with highly satisfactory results. The coefficient of restitution was evaluated for a range of velocity factors of 4 to 100 volts. Slight deviations in the coefficient of restitution were noted when the velocity was below 3 volts.

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